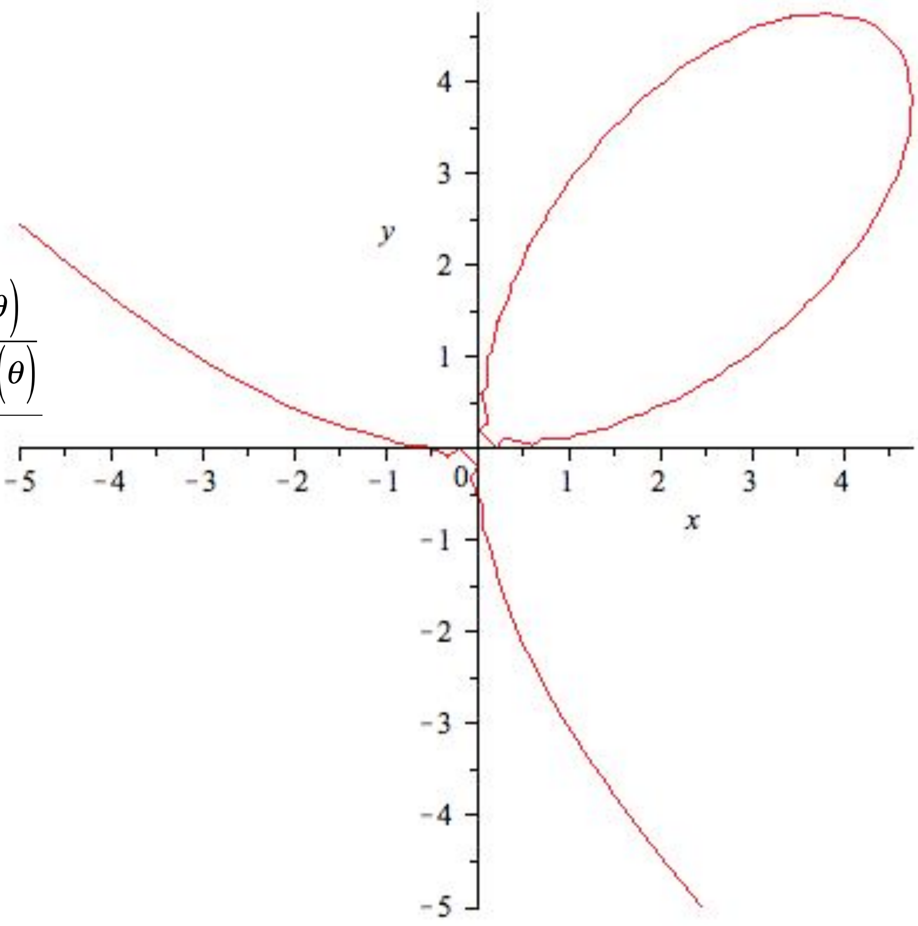
Implicit Differentiation

$x^3 + y^3 - 9xy = 0$

$$r = \frac{9\sin(\theta)\cos(\theta)}{\sin^3(\theta) + \cos^3(\theta)}$$

$$x = \frac{9T}{1+T^3}$$

$$y = \frac{9T^2}{1+T^3}$$



-The graph above has a well-defined slope at nearly every point because it is the union of $y = f_1(x)$, $y = f_2(x)$, $y = f_3(x)$ which are differentiable except at 0 and A.

-We need new techniques to solve for the slopes.
-To do this we need to

-Differentiate both sides of the equation with respect to x.
-Solve for $\frac{dy}{dx}$ in terms of x and y <u>together</u>!

-The process we use to find $\frac{dy}{dx}$ is called <u>implicit differentiation</u> because we don't know $y = f_1(x)$, $y = f_2(x)$, $y = f_3(x)$

Example

Find $\frac{dy}{dx}$ if $y^2 = x$

-To solve, differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.

$y^2 = x$

$2y\frac{dy}{dx} = 1$ $\quad\quad\quad$ $\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2) \bullet \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{2y}$

-This is helpful in this case.

$y^2 = x$

$y = \sqrt{x}$

-This graph has 2 slopes for $x = 4$ one at $(4,2)$ and $(4,-2)$.

-Here our derivative is in terms of y so we can use y_1 and y_2 to find BOTH derivatives.

$\frac{1}{2y_1} = \frac{1}{4}$ $\quad\quad\quad$ $\frac{1}{2y_2} = -\frac{1}{4}$

Example-Finding Slope of a Circle

Find the slope of the circle $x^2 + y^2 = 25$ at $(3,-4)$

-To graph this we would need to solve for *y*:

$y_1 = \sqrt{25 - x^2}$ $\quad\quad$ $y_2 = -\sqrt{25 - x^2}$

-Since $(3,-4)$ is on y_2 we could calculate explicitly

$$\frac{dy_2}{dx} = -\left(25-x^2\right)^{1/2} = \frac{-1}{\sqrt{25-x^2}}\left(2x\right)$$
$$= \left.\frac{-2x}{\sqrt{25-x^2}}\right|_{x=3} = \frac{-6}{2\sqrt{25-9}} = \frac{3}{4}$$

-This can be done even easier using implicit differentiation and will work on the entire circle.

$$\frac{d}{dx}\left(x^2+y^2\right) = \frac{d}{dx}\left(25\right)$$

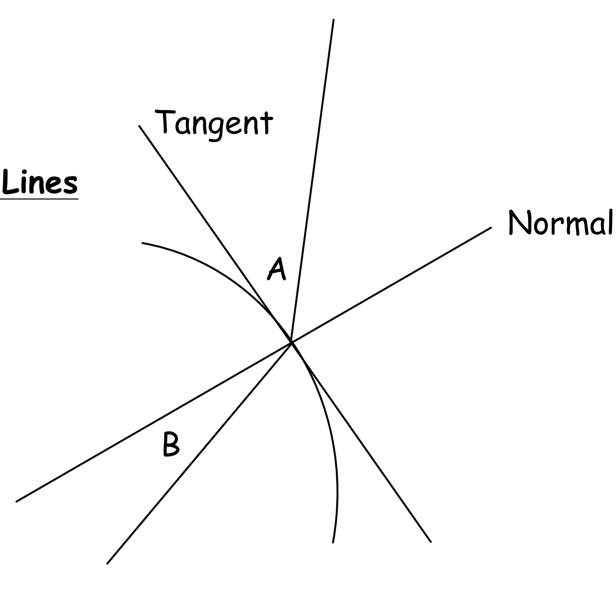
$$2x+2y\left(\frac{dy}{dx}\right) = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

At $(3,-4)$

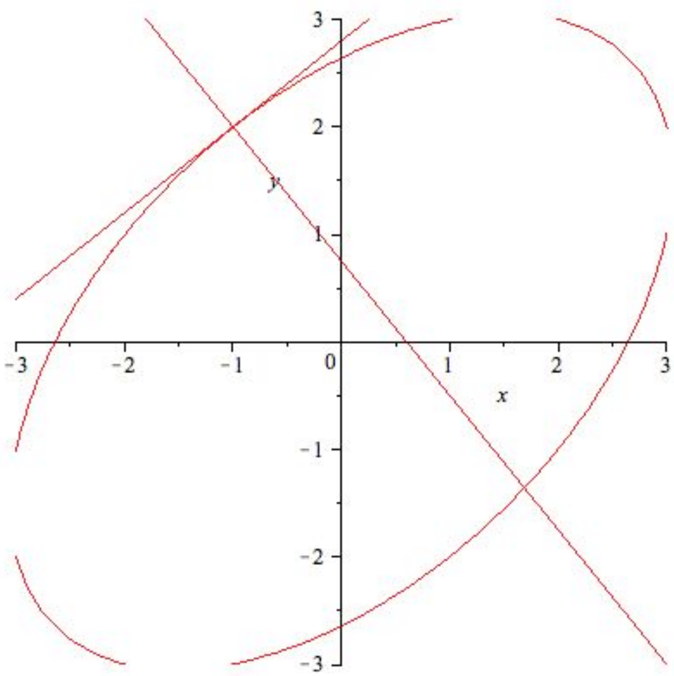
$$= \frac{-3}{-4} = \frac{3}{4}$$

Lenses, Tangents, and Normal Lines



Example

-Find the tangent and normal to the ellipse $x^2 - xy + y^2 = 7$ at $(-1,2)$.



SOLUTION:

$$\frac{d}{dx}\left(x^2\right) - \frac{d}{dx}\left(xy\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(7\right)$$

$$2x - \left(x\frac{dy}{dx} + y\left(1\right)\right) + 2y\frac{dy}{dx} = 0$$ 'Treat xy as a product

$$\left(2y - x\right)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{2y - x}$$

Evaluate at $x=-1, y=2$

$$\frac{dy}{dx}=\frac{2-2\left(-1\right)}{2\left(2\right)-\left(-1\right)}=\frac{4}{5}$$

Tangent at $x=-1, y=2$

$$y-2=\frac{4}{5}\left(x+1\right)$$

$$y-2=\frac{4}{5}x+\frac{4}{5}$$

$$y=\frac{4}{5}x+\frac{14}{5}$$

Normal at $x=-1, y=2$

$$y-2=-\frac{5}{4}\left(x+1\right)$$

$$y-2=-\frac{5}{4}x-\frac{5}{4}$$

$$y=-\frac{5}{4}x+\frac{3}{4}$$

Example

-Show the slope is defined everywhere on $2y=x^2+\sin y$

$$2\frac{dy}{dx}=2x+\cos y\frac{dy}{dx}$$

Will $2-\cos y$ ever equal zero?

NO!!

$$2\frac{dy}{dx}-\cos y\frac{dy}{dx}=2x$$

$$\left(2-\cos y\right)\frac{dy}{dx}=2x$$

$$\frac{dy}{dx}=\frac{2x}{2-\cos y}$$

Example-Second Derivative Implicitly

Find $\frac{d^2y}{dx^2}$ if $2x^3-3y^2=0$.

Note: $y'=\frac{dy}{dx}$

$$6x^2-6yy'=0$$

$$x^2-yy'=0$$

$$y'=\frac{x^2}{y}$$

To find y''.

$$y''=\frac{2xy-x^2y'}{y^2}=\frac{2x}{y}-\frac{x^2y'}{y^2}$$

Don't forget we already solved $y'=\frac{x^2}{y}$

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right)$$

$$= \frac{2x}{y} - \frac{x^4}{y^3}$$